

6. Gamma Function and Related Functions

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¹ National Bureau of Standards.

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6. Gamma Function and Related Functions

Mathematical Properties

6.1. Gamma (Factorial) Function

Euler's Integral

$$\begin{aligned} 6.1.1 \quad \Gamma(z) &= \int_0^{\infty} t^{z-1} e^{-t} dt \quad (\Re z > 0) \\ &= k^z \int_0^{\infty} t^{z-1} e^{-kt} dt \quad (\Re z > 0, \Re k > 0) \end{aligned}$$

Euler's Formula

$$6.1.2 \quad \Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{z(z+1) \dots (z+n)} \quad (z \neq 0, -1, -2, \dots)$$

Euler's Infinite Product

$$\begin{aligned} 6.1.3 \quad \frac{1}{\Gamma(z)} &= z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n} \right) e^{-z/n} \right] \quad (|z| < \infty) \\ \gamma &= \lim_{m \rightarrow \infty} \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{m} - \ln m \right] \\ &= .57721\ 56649 \dots \end{aligned}$$

γ is known as Euler's constant and is given to 25 decimal places in chapter 1. $\Gamma(z)$ is single valued and analytic over the entire complex plane, save for the points $z = -n$ ($n = 0, 1, 2, \dots$) where it possesses simple poles with residue $(-1)^n/n!$. Its reciprocal $1/\Gamma(z)$ is an entire function possessing simple zeros at the points $z = -n$ ($n = 0, 1, 2, \dots$).

Hankel's Contour Integral

$$6.1.4 \quad \frac{1}{\Gamma(z)} = \frac{i}{2\pi} \int_C (-t)^{-z} e^{-t} dt \quad (|z| < \infty)$$

The path of integration C starts at $+\infty$ on the real axis, circles the origin in the counterclockwise direction and returns to the starting point.

Factorial and Π Notations

$$6.1.5 \quad \Pi(z) = z! = \Gamma(z+1)$$

Integer Values

$$6.1.6 \quad \Gamma(n+1) = 1 \cdot 2 \cdot 3 \dots (n-1)n = n!$$

6.1.7

$$\lim_{z \rightarrow n} \frac{1}{\Gamma(-z)} = 0 = \frac{1}{(-n-1)!} \quad (n = 0, 1, 2, \dots)$$

Fractional Values

$$6.1.8 \quad \Gamma\left(\frac{1}{2}\right) = 2 \int_0^{\infty} e^{-t^2} dt = \pi^{\frac{1}{2}} = 1.77245\ 38509 \dots = \left(-\frac{1}{2}\right)!$$

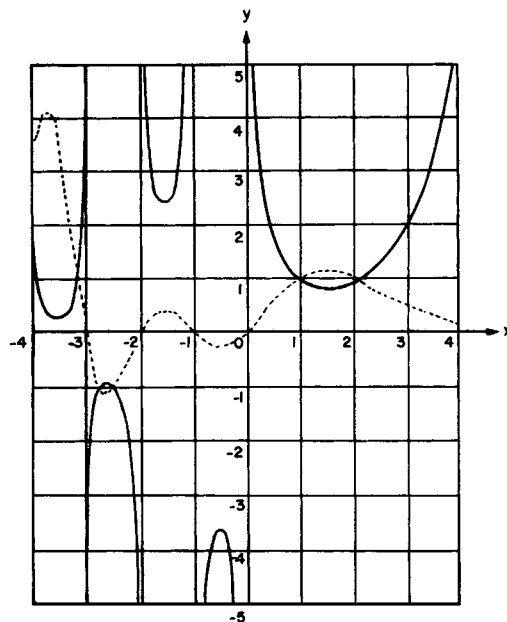


FIGURE 6.1. Gamma function. *

—, $y = \Gamma(x)$, - - -, $y = 1/\Gamma(x)$

$$6.1.9 \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\pi^{\frac{1}{2}} = .88622\ 69254 \dots = \left(\frac{1}{2}\right)!$$

$$6.1.10 \quad \Gamma\left(n + \frac{1}{4}\right) = \frac{1 \cdot 5 \cdot 9 \cdot 13 \dots (4n-3)}{4^n} \Gamma\left(\frac{1}{4}\right)$$

$$\Gamma\left(\frac{1}{4}\right) = 3.62560\ 99082 \dots$$

$$6.1.11 \quad \Gamma\left(n + \frac{1}{3}\right) = \frac{1 \cdot 4 \cdot 7 \cdot 10 \dots (3n-2)}{3^n} \Gamma\left(\frac{1}{3}\right)$$

$$\Gamma\left(\frac{1}{3}\right) = 2.67893\ 85347 \dots$$

$$6.1.12 \quad \Gamma\left(n + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots (2n-1)}{2^n} \Gamma\left(\frac{1}{2}\right)$$

$$6.1.13 \quad \Gamma\left(n + \frac{2}{3}\right) = \frac{2 \cdot 5 \cdot 8 \cdot 11 \dots (3n-1)}{3^n} \Gamma\left(\frac{2}{3}\right)$$

$$\Gamma\left(\frac{2}{3}\right) = 1.35411\ 79394 \dots$$

$$6.1.14 \quad \Gamma\left(n + \frac{3}{4}\right) = \frac{3 \cdot 7 \cdot 11 \cdot 15 \dots (4n-1)}{4^n} \Gamma\left(\frac{3}{4}\right)$$

$$\Gamma\left(\frac{3}{4}\right) = 1.22541\ 67024 \dots$$

*See page II.

Recurrence Formulas

$$6.1.15 \quad \Gamma(z+1) = z\Gamma(z) = z! = z(z-1)!$$

$$6.1.16$$

$$\begin{aligned} \Gamma(n+z) &= (n-1+z)(n-2+z) \dots (1+z)\Gamma(1+z) \\ &= (n-1+z)! \\ &= (n-1+z)(n-2+z) \dots (1+z)z! \end{aligned}$$

Reflection Formula

$$6.1.17 \quad \Gamma(z)\Gamma(1-z) = -z\Gamma(-z)\Gamma(z) = \pi \csc \pi z$$

$$= \int_0^{\infty} \frac{t^{z-1}}{1+t} dt \quad (0 < \Re z < 1)$$

Duplication Formula

$$6.1.18 \quad \Gamma(2z) = (2\pi)^{-\frac{1}{2}} 2^{2z-1} \Gamma(z) \Gamma(z+\frac{1}{2})$$

Triplcation Formula

$$6.1.19 \quad \Gamma(3z) = (2\pi)^{-1} 3^{3z-1} \Gamma(z) \Gamma(z+\frac{1}{3}) \Gamma(z+\frac{2}{3})$$

Gauss' Multiplication Formula

$$6.1.20 \quad \Gamma(nz) = (2\pi)^{\frac{1}{2}(1-n)} n^{nz-\frac{1}{2}} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right)$$

Binomial Coefficient

$$6.1.21 \quad \binom{z}{w} = \frac{z!}{w!(z-w)!} = \frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}$$

Pochhammer's Symbol

$$6.1.22$$

$$(z)_0 = 1,$$

$$(z)_n = z(z+1)(z+2) \dots (z+n-1) = \frac{\Gamma(z+n)}{\Gamma(z)}$$

Gamma Function in the Complex Plane

$$6.1.23 \quad \Gamma(\bar{z}) = \overline{\Gamma(z)}; \ln \Gamma(\bar{z}) = \overline{\ln \Gamma(z)}$$

$$6.1.24 \quad \arg \Gamma(z+1) = \arg \Gamma(z) + \arctan \frac{y}{x}$$

$$6.1.25 \quad \left| \frac{\Gamma(x+iy)}{\Gamma(x)} \right|^2 = \prod_{n=0}^{\infty} \left[1 + \frac{y^2}{(x+n)^2} \right]^{-1}$$

$$6.1.26 \quad |\Gamma(x+iy)| \leq |\Gamma(x)|$$

$$6.1.27$$

$$\arg \Gamma(x+iy) = y\psi(x) + \sum_{n=0}^{\infty} \left(\frac{y}{x+n} - \arctan \frac{y}{x+n} \right)$$

$$(x+iy \neq 0, -1, -2, \dots)$$

where

$$\psi(z) = \Gamma'(z)/\Gamma(z)$$

$$6.1.28 \quad \Gamma(1+iy) = iy \Gamma(iy)$$

$$6.1.29 \quad \Gamma(iy)\Gamma(-iy) = |\Gamma(iy)|^2 = \frac{\pi}{y \sinh \pi y}$$

$$6.1.30 \quad \Gamma(\frac{1}{2}+iy)\Gamma(\frac{1}{2}-iy) = |\Gamma(\frac{1}{2}+iy)|^2 = \frac{\pi}{\cosh \pi y}$$

$$6.1.31 \quad \Gamma(1+iy)\Gamma(1-iy) = |\Gamma(1+iy)|^2 = \frac{\pi y}{\sinh \pi y}$$

$$6.1.32 \quad \Gamma(\frac{1}{4}+iy)\Gamma(\frac{3}{4}-iy) = \frac{\pi\sqrt{2}}{\cosh \pi y + i \sinh \pi y}$$

Power Series

$$6.1.33$$

$$\begin{aligned} \ln \Gamma(1+z) &= -\ln(1+z) + z(1-\gamma) \\ &+ \sum_{n=2}^{\infty} (-1)^n [\zeta(n)-1] z^n/n \quad (|z| < 2) \end{aligned}$$

$\zeta(n)$ is the Riemann Zeta Function (see chapter 23).

Series Expansion² for $1/\Gamma(z)$

$$6.1.34 \quad \frac{1}{\Gamma(z)} = \sum_{k=1}^{\infty} c_k z^k \quad (|z| < \infty)$$

k	c_k
1	1.00000 00000 000000
2	0.57721 56649 015329
3	-0.65587 80715 202538
4	-0.04200 26350 340952
5	0.16653 86113 822915
6	-0.04219 77345 555443
7	-0.00962 19715 278770
8	0.00721 89432 466630
9	-0.00116 51675 918591
10	-0.00021 52416 741149
11	0.00012 80502 823882
12	-0.00002 01348 547807
13	-0.00000 12504 934821
14	0.00000 11330 272320
15	-0.00000 02056 338417
16	0.00000 00061 160950
17	0.00000 00050 020075
18	-0.00000 00011 812746
19	0.00000 00001 043427
20	0.00000 00000 077823
21	-0.00000 00000 036968
22	0.00000 00000 005100
23	-0.00000 00000 000206
24	-0.00000 00000 000054
25	0.00000 00000 000014
26	0.00000 00000 000001

² The coefficients c_k are from H. T. Davis, Tables of higher mathematical functions, 2 vols., Principia Press, Bloomington, Ind., 1933, 1935 (with permission); with corrections due to H. E. Salzer.

Polynomial Approximations³6.1.35 $0 \leq x \leq 1$

$$\Gamma(x+1) = x! = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \epsilon(x)$$

$$|\epsilon(x)| \leq 5 \times 10^{-6}$$

$$\begin{array}{ll} a_1 = -.57486\ 46 & a_4 = .42455\ 49 \\ a_2 = .95123\ 63 & a_5 = -.10106\ 78 \\ a_3 = -.69985\ 88 & \end{array}$$

6.1.36 $0 \leq x \leq 1$

$$\Gamma(x+1) = x! = 1 + b_1x + b_2x^2 + \dots + b_8x^8 + \epsilon(x)$$

$$|\epsilon(x)| \leq 3 \times 10^{-7}$$

$$\begin{array}{ll} b_1 = -.57719\ 1652 & b_5 = -.75670\ 4078 \\ b_2 = .98820\ 5891 & b_6 = .48219\ 9394 \\ b_3 = -.89705\ 6937 & b_7 = -.19352\ 7818 \\ b_4 = .91820\ 6857 & b_8 = .03586\ 8343 \end{array}$$

Stirling's Formula

6.1.37

$$\Gamma(z) \sim e^{-z} z^{z-\frac{1}{2}} (2\pi)^{\frac{1}{2}} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + \dots \right] \quad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

6.1.38

$$x! = \sqrt{2\pi} x^{x+\frac{1}{2}} \exp\left(-x + \frac{\theta}{12x}\right) \quad (x > 0, 0 < \theta < 1)$$

Asymptotic Formulas

6.1.39

$$\Gamma(az+b) \sim \sqrt{2\pi} e^{-az} (az)^{az+b-\frac{1}{2}} \quad (|\arg z| < \pi, a > 0)$$

6.1.40

$$\ln \Gamma(z) \sim (z - \frac{1}{2}) \ln z - z + \frac{1}{2} \ln(2\pi) + \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)z^{2m-1}} \quad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

For B_n see chapter 23

6.1.41

$$\ln \Gamma(z) \sim (z - \frac{1}{2}) \ln z - z + \frac{1}{2} \ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \frac{1}{1680z^7} + \dots \quad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

³ From C. Hastings, Jr., Approximations for digital computers, Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

Error Term for Asymptotic Expansion

6.1.42

If

$$R_n(z) = \ln \Gamma(z) - (z - \frac{1}{2}) \ln z + z - \frac{1}{2} \ln(2\pi)$$

$$- \sum_{m=1}^n \frac{B_{2m}}{2m(2m-1)z^{2m-1}}$$

then

$$|R_n(z)| \leq \frac{|B_{2n+2}|K(z)}{(2n+1)(2n+2)|z|^{2n+1}}$$

where

$$K(z) = \text{upper bound}_{u \geq 0} |z^2/(u^2+z^2)|$$

For z real and positive, R_n is less in absolute value than the first term neglected and has the same sign.

6.1.43

$$\begin{aligned} \mathcal{R} \ln \Gamma(iy) &= \mathcal{R} \ln \Gamma(-iy) \\ &= \frac{1}{2} \ln \left(\frac{\pi}{y \sinh \pi y} \right) \\ &\sim \frac{1}{2} \ln(2\pi) - \frac{1}{2} \pi y - \frac{1}{2} \ln y, \quad (y \rightarrow +\infty) \end{aligned}$$

6.1.44

$$\begin{aligned} \mathcal{I} \ln \Gamma(iy) &= \arg \Gamma(iy) = -\arg \Gamma(-iy) \\ &= -\mathcal{I} \ln \Gamma(-iy) \\ &\sim y \ln y - y - \frac{1}{2} \pi - \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_{2n}}{(2n-1)(2n)y^{2n-1}} \quad (y \rightarrow +\infty) \end{aligned}$$

$$6.1.45 \quad \lim_{|y| \rightarrow \infty} (2\pi)^{-\frac{1}{2}} |\Gamma(x+iy)| e^{\frac{1}{2}\pi|y|} |y|^{\frac{1}{2}-x} = 1$$

$$6.1.46 \quad \lim_{n \rightarrow \infty} n^{b-a} \frac{\Gamma(n+a)}{\Gamma(n+b)} = 1$$

6.1.47

$$\begin{aligned} z^{b-a} \frac{\Gamma(z+a)}{\Gamma(z+b)} &\sim 1 + \frac{(a-b)(a+b-1)}{2z} \\ &\quad + \frac{1}{12} \binom{a-b}{2} (3(a+b-1)^2 - a + b - 1) \frac{1}{z^2} + \dots \end{aligned}$$

as $z \rightarrow \infty$ along any curve joining $z=0$ and $z=\infty$, providing $z \neq -a, -a-1, \dots; z \neq -b, -b-1, \dots$

Continued Fraction

6.1.48

$$\ln \Gamma(z) + z - (z - \frac{1}{2}) \ln z - \frac{1}{2} \ln(2\pi)$$

$$= \frac{a_0}{z+} \frac{a_1}{z+} \frac{a_2}{z+} \frac{a_3}{z+} \frac{a_4}{z+} \frac{a_5}{z+} \dots \quad (\Re z > 0)$$

$$a_0 = \frac{1}{12}, a_1 = \frac{1}{30}, a_2 = \frac{53}{210}, a_3 = \frac{195}{371},$$

$$a_4 = \frac{22999}{22737}, a_5 = \frac{29944523}{19733142}, a_6 = \frac{109535241009}{48264275462}$$

Wallis' Formula⁴

6.1.49

$$\frac{2}{\pi} \int_0^{\pi/2} \left(\frac{\sin}{\cos} \right)^{2n} x dx = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n)}$$

$$= \frac{(2n)!}{2^{2n} (n!)^2} = \frac{1}{2^{2n}} \binom{2n}{n} = \frac{\Gamma(n+\frac{1}{2})}{\pi^{\frac{1}{2}} \Gamma(n+1)}$$

$$\sim \frac{1}{\pi^{\frac{1}{2}} n^{\frac{1}{2}}} \left[1 - \frac{1}{8n} + \frac{1}{128n^2} - \dots \right]$$

$$(n \rightarrow \infty)$$

Some Definite Integrals

6.1.50

$$\ln \Gamma(z) = \int_0^\infty \left[(z-1) e^{-t} - \frac{e^{-t} - e^{-zt}}{1 - e^{-t}} \right] \frac{dt}{t} \quad (\Re z > 0)$$

$$= (z - \frac{1}{2}) \ln z - z + \frac{1}{2} \ln 2\pi$$

$$+ 2 \int_0^\infty \frac{\arctan(t/z)}{e^{2\pi t} - 1} dt \quad (\Re z > 0)$$

6.2. Beta Function

6.2.1

$$B(z, w) = \int_0^1 t^{z-1} (1-t)^{w-1} dt = \int_0^\infty \frac{t^{z-1}}{(1+t)^{z+w}} dt$$

$$= 2 \int_0^{\pi/2} (\sin t)^{2z-1} (\cos t)^{2w-1} dt$$

$$(\Re z > 0, \Re w > 0)$$

$$6.2.2 \quad B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = B(w, z)$$

6.3. Psi (Digamma) Function⁵

$$6.3.1 \quad \psi(z) = d[\ln \Gamma(z)]/dz = \Gamma'(z)/\Gamma(z)$$

⁴ Some authors employ the special double factorial notation as follows:

$$(2n)!! = 2 \cdot 4 \cdot 6 \dots (2n) = 2^n n!$$

$$(2n-1)!! = 1 \cdot 3 \cdot 5 \dots (2n-1) = \pi^{-\frac{1}{2}} 2^n \Gamma(n + \frac{1}{2})$$

⁵ Some authors write $\psi(z) = \frac{d}{dz} \ln \Gamma(z+1)$ and similarly for the polygamma functions.

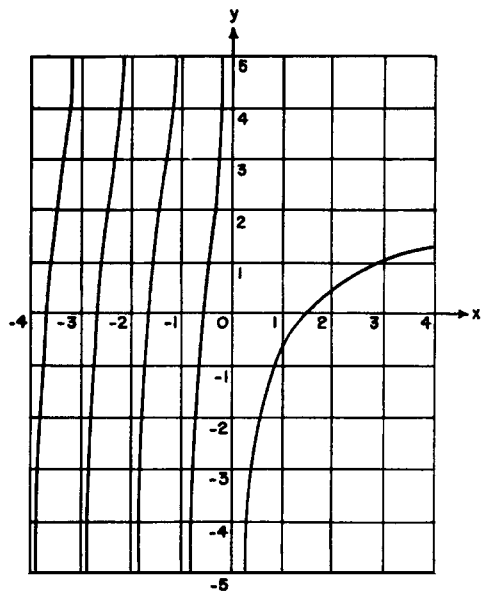


FIGURE 6.2. Psi function.

$$y = \psi(x) = d \ln \Gamma(x)/dx$$

Integer Values

$$6.3.2 \quad \psi(1) = -\gamma, \psi(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1} \quad (n \geq 2)$$

Fractional Values

6.3.3

$$\psi(\frac{1}{2}) = -\gamma - 2 \ln 2 = -1.96351 00260 21423 \dots$$

6.3.4

$$\psi(n + \frac{1}{2}) = -\gamma - 2 \ln 2 + 2 \left(1 + \frac{1}{3} + \dots + \frac{1}{2n-1} \right)$$

$$(n \geq 1)$$

Recurrence Formulas

$$6.3.5 \quad \psi(z+1) = \psi(z) + \frac{1}{z}$$

6.3.6

$$\psi(n+z) = \frac{1}{(n-1)+z} + \frac{1}{(n-2)+z} + \dots$$

$$+ \frac{1}{2+z} + \frac{1}{1+z} + \psi(1+z)$$

Reflection Formula

$$6.3.7 \quad \psi(1-z) = \psi(z) + \pi \cot \pi z$$

Duplication Formula

$$6.3.8 \quad \psi(2z) = \frac{1}{2}\psi(z) + \frac{1}{2}\psi\left(z + \frac{1}{2}\right) + \ln 2$$

Psi Function in the Complex Plane

$$6.3.9 \quad \psi(\bar{z}) = \overline{\psi(z)}$$

$$6.3.10$$

$$\Re \psi(iy) = \Re \psi(-iy) = \Re \psi(1+iy) = \Re \psi(1-iy)$$

$$6.3.11 \quad \Im \psi(iy) = \frac{1}{2}y^{-1} + \frac{1}{2}\pi \coth \pi y$$

$$6.3.12 \quad \Im \psi\left(\frac{1}{2} + iy\right) = \frac{1}{2}\pi \tanh \pi y$$

$$6.3.13 \quad \Im \psi(1+iy) = -\frac{1}{2y} + \frac{1}{2}\pi \coth \pi y$$

$$= y \sum_{n=1}^{\infty} (n^2 + y^2)^{-1}$$

Series Expansions

$$6.3.14 \quad \psi(1+z) = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n) z^{n-1} \quad (|z| < 1)$$

$$6.3.15$$

$$\psi(1+z) = \frac{1}{2}z^{-1} - \frac{1}{2}\pi \cot \pi z - (1-z^2)^{-1} + 1 - \gamma$$

$$- \sum_{n=1}^{\infty} [\zeta(2n+1) - 1] z^{2n} \quad (|z| < 2)$$

$$6.3.16$$

$$\psi(1+z) = -\gamma + \sum_{n=1}^{\infty} \frac{z}{n(n+z)} \quad (z \neq -1, -2, -3, \dots)$$

$$6.3.17$$

$$\Re \psi(1+iy) = 1 - \gamma - \frac{1}{1+y^2}$$

$$+ \sum_{n=1}^{\infty} (-1)^{n+1} [\zeta(2n+1) - 1] y^{2n}$$

$$(|y| < 2)$$

$$= -\gamma + y^2 \sum_{n=1}^{\infty} n^{-1} (n^2 + y^2)^{-1}$$

$$(-\infty < y < \infty)$$

Asymptotic Formulas

$$6.3.18$$

$$\psi(z) \sim \ln z - \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nz^{2n}}$$

$$= \ln z - \frac{1}{2z} - \frac{1}{12z^3} + \frac{1}{120z^5} - \frac{1}{252z^7} + \dots$$

$$(z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

6.3.19

$$\Re \psi(1+iy) \sim \ln y + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_{2n}}{2ny^{2n}}$$

$$= \ln y + \frac{1}{12y^2} + \frac{1}{120y^4} + \frac{1}{252y^6} + \dots$$

$$(y \rightarrow \infty)$$

Extrema^a of $\Gamma(x)$ — Zeros of $\psi(x)$

$$\Gamma'(x_n) = \psi(x_n) = 0$$

n	x_n	$\Gamma'(x_n)$
0	+1.462	+0.886
1	-0.504	-3.545
2	-1.573	+2.302
3	-2.611	-0.888
4	-3.635	+0.245
5	-4.653	-0.053
6	-5.667	+0.009
7	-6.678	-0.001

$$x_0 = 1.46163 \quad 21449 \quad 68362$$

$$\Gamma(x_0) = .88560 \quad 31944 \quad 10889$$

$$6.3.20 \quad x_n = -n + (\ln n)^{-1} + o[(\ln n)^{-2}]$$

Definite Integrals

$$6.3.21$$

$$\psi(z) = \int_0^{\infty} \left[\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}} \right] dt \quad (\Re z > 0)$$

$$= \int_0^{\infty} \left[e^{-t} - \frac{1}{(1+t)^z} \right] \frac{dt}{t}$$

$$= \ln z - \frac{1}{2z} - 2 \int_0^{\infty} \frac{t dt}{(t^2 + z^2)(e^{2\pi t} - 1)}$$

$$\left(|\arg z| < \frac{\pi}{2} \right)$$

$$6.3.22$$

$$\psi(z) + \gamma = \int_0^{\infty} \frac{e^{-t} - e^{-zt}}{1 - e^{-t}} dt = \int_0^1 \frac{1 - t^{z-1}}{1 - t} dt$$

$$\gamma = \int_0^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{te^t} \right) dt$$

$$= \int_0^{\infty} \left(\frac{1}{1+t} - e^{-t} \right) \frac{dt}{t}$$

^a From W. Sibagaki, Theory and applications of the gamma function, Iwanami Syoten, Tokyo, Japan, 1952 (with permission).

6.4. Polygamma Functions⁷

6.4.1

$$\psi^{(n)}(z) = \frac{d^n}{dz^n} \psi(z) = \frac{d^{n+1}}{dz^{n+1}} \ln \Gamma(z) \quad (n=1, 2, 3, \dots)$$

$$* \quad = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-zt}}{1 - e^{-t}} dt \quad (\Re z > 0)$$

$\psi^{(n)}(z)$, ($n=0, 1, \dots$), is a single valued analytic function over the entire complex plane save at the points $z = -m$ ($m=0, 1, 2, \dots$) where it possesses poles of order $(n+1)$.

Integer Values

6.4.2

$$\psi^{(n)}(1) = (-1)^{n+1} n! \zeta(n+1) \quad (n=1, 2, 3, \dots)$$

6.4.3

$$\psi^{(m)}(n+1) = (-1)^m m! \left[-\zeta(m+1) + 1 + \frac{1}{2^{m+1}} + \dots + \frac{1}{n^{m+1}} \right]$$

Fractional Values

6.4.4

$$\psi^{(n)}\left(\frac{1}{2}\right) = (-1)^{n+1} n! (2^{n+1} - 1) \zeta(n+1) \quad (n=1, 2, \dots)$$

$$6.4.5 \quad \psi'\left(n + \frac{1}{2}\right) = \frac{1}{2} \pi^2 - 4 \sum_{k=1}^n (2k-1)^{-2}$$

Recurrence Formula

$$6.4.6 \quad \psi^{(n)}(z+1) = \psi^{(n)}(z) + (-1)^n n! z^{-n-1}$$

Reflection Formula

6.4.7

$$\psi^{(n)}(1-z) + (-1)^{n+1} \psi^{(n)}(z) = (-1)^n \pi \frac{d^n}{dz^n} \cot \pi z$$

Multiplication Formula

6.4.8

$$* \quad \psi^{(n)}(mz) = \delta \ln m + \frac{1}{m^{n+1}} \sum_{k=0}^{m-1} \psi^{(n)}\left(z + \frac{k}{m}\right)$$

$$\delta = 1, \quad n = 0$$

$$\delta = 0, \quad n > 0$$

⁷ ψ' is known as the trigamma function. ψ'' , $\psi^{(3)}$, $\psi^{(4)}$ are the tetra-, penta-, and hexagramma functions respectively. Some authors write $\psi(z) = d[\ln \Gamma(z+1)]/dz$, and similarly for the polygamma functions.

* See page 11.

Series Expansions

6.4.9

$$\psi^{(n)}(1+z) = (-1)^{n+1} \left[n! \zeta(n+1) - \frac{(n+1)!}{1!} \zeta(n+2)z + \frac{(n+2)!}{2!} \zeta(n+3)z^2 - \dots \right] \quad (|z| < 1)$$

6.4.10

$$\psi^{(n)}(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} (z+k)^{-n-1} \quad (z \neq 0, -1, -2, \dots)$$

Asymptotic Formulas

6.4.11

$$\psi^{(n)}(z) \sim (-1)^{n-1} \left[\frac{(n-1)!}{z^n} + \frac{n!}{2z^{n+1}} + \sum_{k=1}^{\infty} B_{2k} \frac{(2k+n-1)!}{(2k)! z^{2k+n}} \right] \quad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

6.4.12

$$\psi'(z) \sim \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} - \frac{1}{30z^5} + \frac{1}{42z^7} - \frac{1}{30z^9} + \dots \quad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

6.4.13

$$\psi''(z) \sim -\frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{2z^4} + \frac{1}{6z^6} - \frac{1}{6z^8} + \frac{3}{10z^{10}} - \frac{5}{6z^{12}} + \dots \quad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

6.4.14

$$\psi^{(3)}(z) \sim \frac{2}{z^3} + \frac{3}{z^4} + \frac{2}{z^5} - \frac{1}{z^7} + \frac{4}{3z^9} - \frac{3}{11z^{11}} + \frac{10}{z^{13}} - \dots \quad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

6.5. Incomplete Gamma Function
(see also 26.4)

6.5.1

$$P(a, x) = \frac{1}{\Gamma(a)} \int_0^x e^{-t} t^{a-1} dt \quad (\Re a > 0)$$

6.5.2

$$\gamma(a, x) = P(a, x) \Gamma(a) = \int_0^x e^{-t} t^{a-1} dt \quad (\Re a > 0)$$

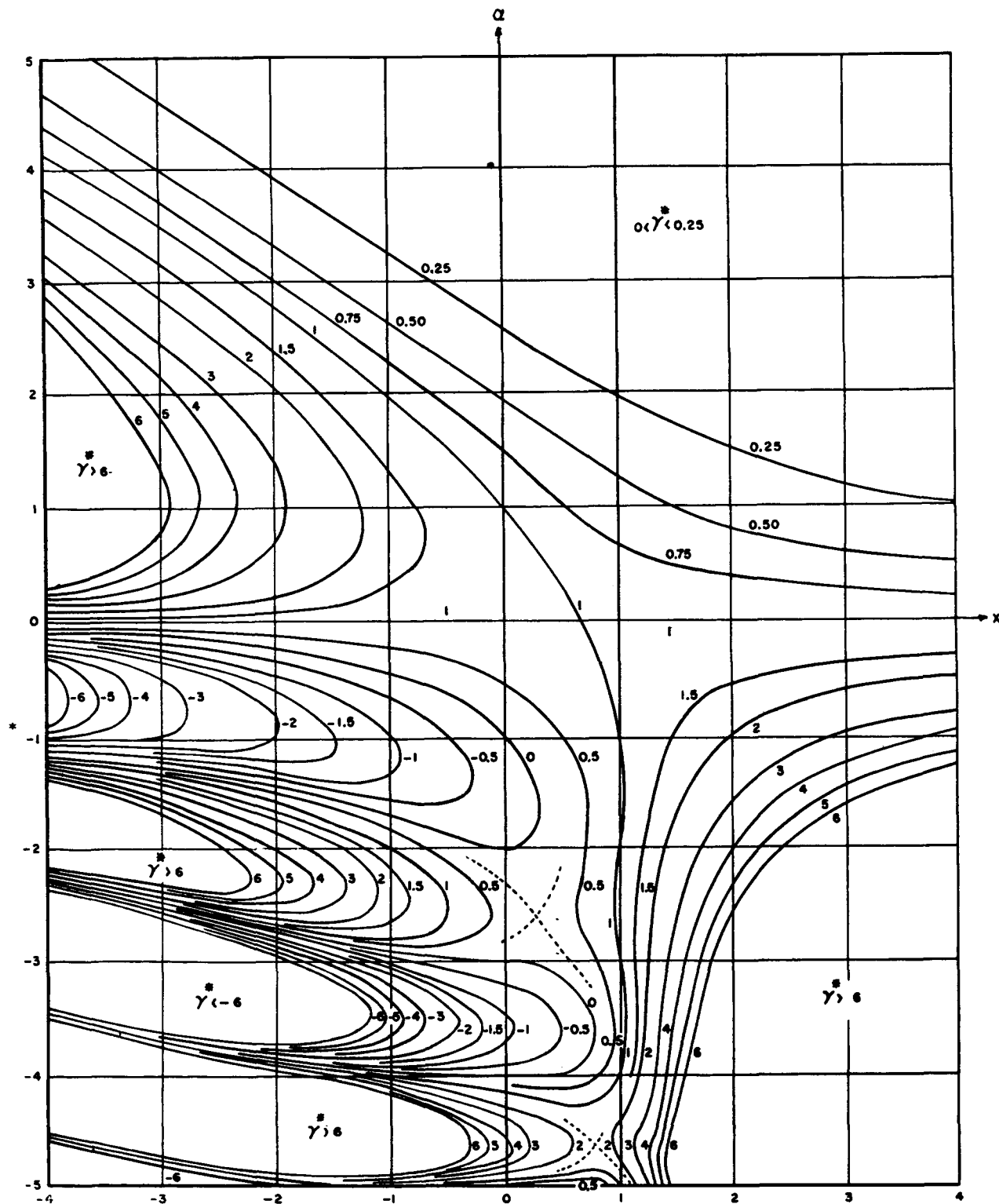
6.5.3

$$\Gamma(a, x) = \Gamma(a) - \gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} dt$$

6.5.4

$$\gamma^*(a, x) = x^{-a} P(a, x) = \frac{x^{-a}}{\Gamma(a)} \gamma(a, x)$$

γ^* is a single valued analytic function of a and x possessing no finite singularities.


 FIGURE 6.3. *Incomplete gamma function.*

$$\gamma^*(a, x) = \frac{x^{-a}}{\Gamma(a)} \int_0^x e^{-t} t^{a-1} dt$$

 From F. G. Tricomi, Sulla funzione gamma incompleta, *Annali di Matematica*, IV, 33, 1950 (with permission).

6.5.5

Probability Integral of the χ^2 -Distribution

$$P(\chi^2|\nu) = \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \int_0^{\chi^2} t^{\frac{\nu}{2}-1} e^{-\frac{1}{2}t} dt$$

6.5.6

(Pearson's Form of the Incomplete Gamma Function)

$$I(u, p) = \frac{1}{\Gamma(p+1)} \int_0^{u\sqrt{p+1}} e^{-t} t^p dt \\ = P(p+1, u\sqrt{p+1})$$

$$6.5.7 \quad C(x, a) = \int_x^\infty t^{a-1} \cos t dt \quad (\Re a < 1)$$

$$6.5.8 \quad S(x, a) = \int_x^\infty t^{a-1} \sin t dt \quad (\Re a < 1)$$

6.5.9

$$E_n(x) = \int_1^\infty e^{-xt} t^{-n} dt = x^{n-1} \Gamma(1-n, x)$$

6.5.10

$$\alpha_n(x) = \int_1^\infty e^{-xt} t^n dt = x^{-n-1} \Gamma(1+n, x)$$

6.5.11

$$e_n(x) = \sum_{j=0}^n \frac{x^j}{j!}$$

Incomplete Gamma Function as a Confluent Hypergeometric Function (see chapter 13)

$$6.5.12 \quad \gamma(a, x) = a^{-1} x^a e^{-x} M(1, 1+a, x) \\ = a^{-1} x^a M(a, 1+a, -x)$$

Special Values

6.5.13

$$P(n, x) = 1 - \left(1 + x + \frac{x^2}{2!} + \dots + \frac{x^{n-1}}{(n-1)!} \right) e^{-x} \\ = 1 - e_{n-1}(x) e^{-x}$$

For relation to the Poisson distribution, see 26.4.

$$6.5.14 \quad \gamma^*(-n, x) = x^n$$

$$6.5.15 \quad \Gamma(0, x) = \int_x^\infty e^{-t} t^{-1} dt = E_1(x)$$

$$6.5.16 \quad \gamma\left(\frac{1}{2}, x^2\right) = 2 \int_0^x e^{-t^2} dt = \sqrt{\pi} \operatorname{erf} x$$

$$6.5.17 \quad \Gamma\left(\frac{1}{2}, x^2\right) = 2 \int_x^\infty e^{-t^2} dt = \sqrt{\pi} \operatorname{erfc} x$$

$$6.5.18 \quad \frac{1}{2} \sqrt{\pi} x \gamma^*\left(\frac{1}{2}, -x^2\right) = \int_0^x e^{t^2} dt$$

$$6.5.19 \quad \Gamma(-n, x) = \frac{(-1)^n}{n!} \left[E_1(x) - e^{-x} \sum_{j=0}^{n-1} \frac{(-1)^j j!}{x^{j+1}} \right]$$

$$6.5.20 \quad \Gamma(a, ix) = e^{\frac{1}{2}\pi ia} [C(x, a) - iS(x, a)]$$

Recurrence Formulas

$$6.5.21 \quad P(a+1, x) = P(a, x) - \frac{x^a e^{-x}}{\Gamma(a+1)}$$

$$6.5.22 \quad \gamma(a+1, x) = a\gamma(a, x) - x^a e^{-x}$$

$$6.5.23 \quad \gamma^*(a-1, x) = x\gamma^*(a, x) + \frac{e^{-x}}{\Gamma(a)}$$

Derivatives and Differential Equations

6.5.24

$$\left(\frac{\partial \gamma^*}{\partial x} \right)_{x=0} = - \int_x^\infty \frac{e^{-t} dt}{t} - \ln x = -E_1(x) - \ln x$$

$$6.5.25 \quad \frac{\partial \gamma(a, x)}{\partial x} = - \frac{\partial \Gamma(a, x)}{\partial x} = x^{a-1} e^{-x}$$

6.5.26

$$\frac{\partial^n}{\partial x^n} [x^{-a} \Gamma(a, x)] = (-1)^n x^{-a-n} \Gamma(a+n, x) \\ (n=0, 1, 2, \dots)$$

6.5.27

$$\frac{\partial^n}{\partial x^n} [e^x x^a \gamma^*(a, x)] = e^x x^{a-n} \gamma^*(a-n, x) \\ (n=0, 1, 2, \dots)$$

$$6.5.28 \quad x \frac{\partial^2 \gamma^*}{\partial x^2} + (a+1+x) \frac{\partial \gamma^*}{\partial x} + a\gamma^* = 0$$

Series Developments

6.5.29

$$\gamma^*(a, z) = e^{-z} \sum_{n=0}^\infty \frac{z^n}{\Gamma(a+n+1)} = \frac{1}{\Gamma(a)} \sum_{n=0}^\infty \frac{(-z)^n}{(a+n)n!} \\ (|z| < \infty)$$

6.5.30

$$\gamma(a, x+y) - \gamma(a, x) = e^{-x} x^{a-1} \sum_{n=0}^{\infty} \frac{(a-1)(a-2) \cdots (a-n)}{x^n} [1 - e^{-y} e_n(y)]$$

$$(|y| < |x|)$$

Continued Fraction

6.5.31

$$\Gamma(a, x) = e^{-x} x^a \left(\frac{1}{x+1} \frac{1-a}{1+} \frac{1}{x+1} \frac{2-a}{1+} \frac{2}{x+1} \cdots \right)$$

$$(x > 0, |a| < \infty)$$

Asymptotic Expansions

6.5.32

$$\Gamma(a, z) \sim z^{a-1} e^{-z} \left[1 + \frac{a-1}{z} + \frac{(a-1)(a-2)}{z^2} + \cdots \right]$$

$$(z \rightarrow \infty \text{ in } |\arg z| < \frac{3\pi}{2})$$

Suppose $R_n(a, z) = u_{n+1}(a, z) + \cdots$ is the remainder after n terms in this series. Then if a, z are real, we have for $n > a - 2$

$$|R_n(a, z)| \leq |u_{n+1}(a, z)|$$

and sign $R_n(a, z) = \text{sign } u_{n+1}(a, z)$.

$$6.5.33 \quad \gamma(a, z) \sim \sum_{n=0}^{\infty} \frac{(-1)^n z^{a+n}}{(a+n)n!} \quad (a \rightarrow +\infty)$$

$$6.5.34 \quad \lim_{n \rightarrow \infty} \frac{e_n(\alpha n)}{e^{\alpha n}} = \begin{cases} 0 & \text{for } \alpha > 1 \\ \frac{1}{2} & \text{for } \alpha = 1 \\ 1 & \text{for } 0 \leq \alpha < 1 \end{cases}$$

6.5.35

$$\Gamma(z+1, z) \sim e^{-z} z^z \left(\sqrt{\frac{\pi}{2}} z^{\frac{1}{2}} + \frac{2}{3} + \frac{\sqrt{2\pi}}{24} \frac{1}{z^{\frac{1}{2}}} + \cdots \right)$$

$$(z \rightarrow \infty \text{ in } |\arg z| < \frac{1}{2}\pi)$$

Numerical Methods

6.7. Use and Extension of the Tables

Example 1. Compute $\Gamma(6.38)$ to 8S. Using the recurrence relation 6.1.16 and Table 6.1 we have,

$$\Gamma(6.38) = [(5.38)(4.38)(3.38)(2.38)(1.38)]\Gamma(1.38)$$

$$= 232.43671.$$

Example 2. Compute $\ln \Gamma(56.38)$, using Table 6.4 and linear interpolation in f_2 . We have

$$\ln \Gamma(56.38) = (56.38 - \frac{1}{2}) \ln (56.38) - (56.38)$$

$$+ f_2(56.38)$$

Definite Integrals

6.5.36

$$\int_0^{\infty} e^{-at} \Gamma(b, ct) dt = \frac{\Gamma(b)}{a} \left[1 - \frac{c^b}{(a+c)^b} \right]$$

$$(\Re(a+c) > 0, \Re b > -1)$$

6.5.37

$$\int_0^{\infty} t^{a-1} \Gamma(b, t) dt = \frac{\Gamma(a+b)}{a}$$

$$(\Re(a+b) > 0, \Re a > 0)$$

6.6. Incomplete Beta Function

$$6.6.1 \quad B_z(a, b) = \int_0^z t^{a-1} (1-t)^{b-1} dt$$

$$6.6.2 \quad I_z(a, b) = B_z(a, b) / B(a, b)$$

For statistical applications, see 26.5.

Symmetry

$$6.6.3 \quad I_z(a, b) = 1 - I_{1-z}(b, a)$$

Relation to Binomial Expansion

$$6.6.4 \quad I_p(a, n-a+1) = \sum_{j=a}^n \binom{n}{j} p^j (1-p)^{n-j}$$

For binomial distribution, see 26.1.

Recurrence Formulas

$$6.6.5 \quad I_z(a, b) = x I_z(a-1, b) + (1-x) I_z(a, b-1)$$

$$6.6.6 \quad (a+b-ax) I_z(a, b) = a(1-x) I_z(a+1, b-1) + b I_z(a, b+1)$$

$$6.6.7 \quad (a+b) I_z(a, b) = a I_z(a+1, b) + b I_z(a, b+1)$$

Relation to Hypergeometric Function

$$6.6.8 \quad B_z(a, b) = a^{-1} x^a F(a, 1-b; a+1; x)$$

The error of linear interpolation in the table of the function f_2 is smaller than 10^{-7} in this region. Hence, $f_2(56.38) = .92041 \ 67$ and $\ln \Gamma(56.38) = 169.85497 \ 42$.

Direct interpolation in Table 6.4 of $\log_{10} \Gamma(n)$ eliminates the necessity of employing logarithms. However, the error of linear interpolation is .002 so that $\log_{10} \Gamma(n)$ is obtained with a relative error of 10^{-5} .

*See page 11.

Example 3. Compute $\psi(6.38)$ to 8S. Using the recurrence relation 6.3.6 and Table 6.1.

$$\begin{aligned}\psi(6.38) &= \frac{1}{5.38} + \frac{1}{4.38} + \frac{1}{3.38} + \frac{1}{2.38} + \frac{1}{1.38} + \psi(1.38) \\ &= 1.77275\ 59.\end{aligned}$$

Example 4. Compute $\psi(56.38)$. Using Table 6.3 we have $\psi(56.38) = \ln 56.38 - f_3(56.38)$.

The error of linear interpolation in the table of the function f_3 is smaller than 8×10^{-7} in this region. Hence, $f_3(56.38) = .00889\ 53$ and $\psi(56.38) = 4.023219$.

Example 5. Compute $\ln \Gamma(1-i)$. From the reflection principle 6.1.23 and Table 6.7, $\ln \Gamma(1-i) = \overline{\ln \Gamma(1+i)} = -.6509 + .3016i$.

Example 6. Compute $\ln \Gamma(\frac{1}{2} + \frac{1}{2}i)$. Taking the logarithm of the recurrence relation 6.1.15 we have,

$$\begin{aligned}\ln \Gamma(\tfrac{1}{2} + \tfrac{1}{2}i) &= \ln \Gamma(\tfrac{3}{2} + \tfrac{1}{2}i) - \ln(\tfrac{1}{2} + \tfrac{1}{2}i) \\ &= -.23419 + .03467i \\ &\quad - (\tfrac{1}{2} \ln \tfrac{1}{2} + i \arctan 1) \\ &= .11239 - .75073i\end{aligned}$$

The logarithms of complex numbers are found from 4.1.2.

Example 7. Compute $\ln \Gamma(3+7i)$ using the duplication formula 6.1.18. Taking the logarithm of 6.1.18, we have

$$\begin{aligned}-\tfrac{1}{2} \ln 2\pi &= -.91894 \\ (\tfrac{3}{2} + 7i) \ln 2 &= 1.73287 + 4.85203i \\ \ln \Gamma(\tfrac{3}{2} + \tfrac{7}{2}i) &= -3.31598 + 2.32553i \\ \ln \Gamma(2 + \tfrac{7}{2}i) &= -2.66047 + 2.93869i \\ \ln \Gamma(3+7i) &= -5.16252 + 10.11625i\end{aligned}$$

Example 8. Compute $\ln \Gamma(3+7i)$ to 5D using the asymptotic formula 6.1.41. We have

$$\ln(3+7i) = 2.03022\ 15 + 1.16590\ 45i.$$

Then,

$$\begin{aligned}(2.5+7i) \ln(3+7i) &= -3.0857779 + 17.1263119i \\ -(3+7i) &= -3.0000000 - 7.0000000i \\ \tfrac{1}{2} \ln(2\pi) &= .9189385 \\ [12(3+7i)]^{-1} &= .0043103 - .0100575i \\ -[360(3+7i)^3]^{-1} &= .0000059 - .0000022i \\ \ln \Gamma(3+7i) &= -5.16252 + 10.11625i\end{aligned}$$

6.8. Summation of Rational Series by Means of Polygamma Functions

An infinite series whose general term is a rational function of the index may always be reduced to a finite series of psi and polygamma functions. The method will be illustrated by writing the explicit formula when the denominator contains a triple root.

Let the general term of an infinite series have the form

$$u_n = \frac{p(n)}{d_1(n)d_2(n)d_3(n)}$$

where

$$d_1(n) = (n + \alpha_1)(n + \alpha_2) \dots (n + \alpha_m)$$

$$d_2(n) = (n + \beta_1)^2(n + \beta_2)^2 \dots (n + \beta_r)^2$$

$$d_3(n) = (n + \gamma_1)^3(n + \gamma_2)^3 \dots (n + \gamma_s)^3$$

where $p(n)$ is a polynomial of degree $m + 2r + 3s - 2$ at most and where the constants α_i , β_i , and γ_i are distinct. Expand u_n in partial fractions as follows

$$\begin{aligned}u_n &= \sum_{k=1}^m \frac{a_k}{(n + \alpha_k)} + \sum_{k=1}^r \frac{b_{1k}}{(n + \beta_k)} + \frac{b_{2k}}{(n + \beta_k)^2} \\ &\quad + \sum_{k=1}^s \frac{c_{1k}}{(n + \gamma_k)} + \frac{c_{2k}}{(n + \gamma_k)^2} + \frac{c_{3k}}{(n + \gamma_k)^3} \\ &\quad \sum_{k=1}^m a_k + \sum_{k=1}^r b_{1k} + \sum_{k=1}^s c_{1k} = 0.\end{aligned}$$

Then, we may express $\sum_{n=1}^{\infty} u_n$ in terms of the constants appearing in this partial fraction expansion as follows

$$\begin{aligned}\sum_{n=1}^{\infty} u_n &= -\sum_{j=1}^m a_j \psi(1 + \alpha_j) \\ &\quad - \sum_{j=1}^r b_{1j} \psi(1 + \beta_j) + \sum_{j=1}^r b_{2j} \psi'(1 + \beta_j) \\ &\quad - \sum_{j=1}^s c_{1j} \psi(1 + \gamma_j) + \sum_{j=1}^s c_{2j} \psi'(1 + \gamma_j) \\ &\quad - \sum_{j=1}^s \frac{c_{3j}}{2!} \psi''(1 + \gamma_j).\end{aligned}$$

Higher order repetitions in the denominator are handled similarly. If the denominator contains

only simple or double roots, omit the corresponding lines.

Example 9. Find

$$s = \sum_{n=1}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)}.$$

Since

$$\frac{1}{(n+1)(2n+1)(4n+1)} = \frac{\frac{1}{2}}{n+1} - \frac{1}{n+\frac{1}{2}} + \frac{\frac{3}{2}}{n+\frac{3}{4}},$$

we have

$$\alpha_1=1, \alpha_2=\frac{1}{2}, \alpha_3=\frac{1}{4}, a_1=\frac{1}{2}, a_2=-1, a_3=\frac{3}{4}.$$

Thus,

$$s = -\frac{1}{2}\psi(2) + \psi(1\frac{1}{2}) - \frac{3}{4}\psi(1\frac{3}{4}) = .047198.$$

Example 10.

$$\text{Find } s = \sum_{n=1}^{\infty} \frac{1}{n^2(8n+1)^2}.$$

$$\text{Since } \frac{1}{n^2(8n+1)^2} = -\frac{16}{n} + \frac{16}{n+\frac{1}{8}} + \frac{1}{n^2} + \frac{1}{(n+\frac{1}{8})^2},$$

we have,

$$\beta_1=0, \beta_2=\frac{1}{8}, b_{11}=-16, b_{12}=16, b_{21}=1, b_{22}=1.$$

Therefore

$$s = 16\psi(1) - 16\psi(1\frac{1}{8}) + \psi'(1) + \psi'(1\frac{1}{8}) = .013499.$$

Example 11.

$$\text{Evaluate } s = \sum_{n=1}^{\infty} \frac{1}{(n^2+1)(n^2+4)} \quad (\text{see also 6.3.13}).$$

$$\text{We have, } \frac{1}{(n^2+1)(n^2+4)} = \frac{i}{6} \left(\frac{1}{n+i} - \frac{1}{n-i} \right) - \frac{i}{12} \left(\frac{1}{n+2i} - \frac{1}{n-2i} \right).$$

$$\text{Hence, } a_1 = \frac{i}{6}, a_2 = \frac{-i}{6}, a_3 = \frac{-i}{12}, a_4 = \frac{i}{12},$$

$$\alpha_1=i, \alpha_2=-i, \alpha_3=2i, \alpha_4=-2i,$$

and therefore

$$s = \frac{-i}{6} [\psi(1+i) - \psi(1-i)] + \frac{i}{12} [\psi(1+2i) - \psi(1-2i)].$$

By 6.3.9, this reduces to

$$s = \frac{1}{3} \mathcal{J} \psi(1+i) - \frac{1}{6} \mathcal{J} \psi(1+2i).$$

From Table 6.8, $s = .13876$.

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For references to tabular material on the incomplete gamma and incomplete beta functions, see the references in chapter 26.